Effective medium theory for the finite conductivity bed-of-nails structure

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Abstract. We present and illustrate an effective medium theory of the finite conductivity periodic finite wire medium (the bed-of-nails structure). It is constructed using a renormalization approach to obtain the system of equations satisfied by the electromagnetic field and the electric current in the limit of large wavelength-to-period ratio. The model is tested numerically and its domain of validity is characterized using full vector three dimensional finite element calculations. Bed-of-nails structures exhibit large absorption with small reflection while their low fill factor allows considerable freedom to control other characteristics of the metamaterial such as its mechanical, thermal or chemical robustness.

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In recent years, the advent of negative index metamaterials and composites has led to increased interest in effective medium theories. The most popular by far is of course the Lorentz theory approach, it being the most accessible and intuitively appealing. However, the usefulness of Lorentz theory is much diminished when one is interested in materials where the size of objects is much larger than the distances separating them, or materials which are strongly non-local, or in which the scatterers are strongly coupled, leading to behavior of a collective nature [1, 2]. In this work we test and illustrate an effective medium model of the finite conductivity bed-of-nails structure (Fig. 1) based on a two-scale renormalization approach. Instead of letting the wavelength tend to infinity, as customary in effective medium theories, we keep it fixed, and let other geometrical parameters tend to zero. The advantage of this approach is that it leaves us the possibility of keeping *some* of the geometrical parameters fixed (in this case the wire length L), leading to a new type of *partial* homogenization scheme. To put it less formally, we would like to homogenize while keeping the thickness fixed with respect to the wavelength, which prevents us from letting λ tend to infinity, so the only remaining option is to make all the other dimensions (the wire radius r and the period d) tend to zero.

The structure under study is a square biperiodic array of thin wires, of length L, radius r and conductivity σ. We note the period d and the wavelength λ. The renormalization (depicted in Fig. 1) involves a limiting process whereby the three quantities: r, d and $1/\sigma$ tend simultaneously to zero. The parameter governing the limiting process is noted $\eta = d_\eta/d$, the ratio between the renormalized period and the real period. The asymptotics of the other two parameters, σ and r, with respect to η are described by *fixed* parameters κ and γ according to the following relations:

$$\kappa = \frac{\pi r_\eta^2 \sigma_\eta}{\varepsilon_0 \omega d_\eta^2} \tag{1}$$

$$\frac{1}{\gamma} = \eta^2 \log\left(\frac{r_\eta}{d_\eta}\right) \tag{2}$$

where ω is the angular frequency of the electromagnetic field. In other words the conductivity is renormalized inversely to the fill factor, while the radius is renormalized such that the expression $\eta^2 \log(\frac{r_\eta}{d_\eta})$ remains constant.

While these expressions may at first seem obscure, they have simple intuitive interpretations. The first requires the current density to remain constant during the renormalization. Recall that the *static* impedance per unit length of a circular wire is given by

$$Z_{\text{wire}} = \frac{1}{\pi r^2 \sigma} \tag{3}$$

and that the number of wires per unit area is renormalized as $1/d_\eta^2$. The second expression requires the average internal capacitance of the wires to remain constant during renormalization. This feature is known to be essential for their asymptotic behavior (see, for instance Refs. [3, 4]).

The question to be answered now becomes: what happens in the limit $\eta \to 0$? The answer is that the fields converge (in a precise sense described in Ref. [5]) to the

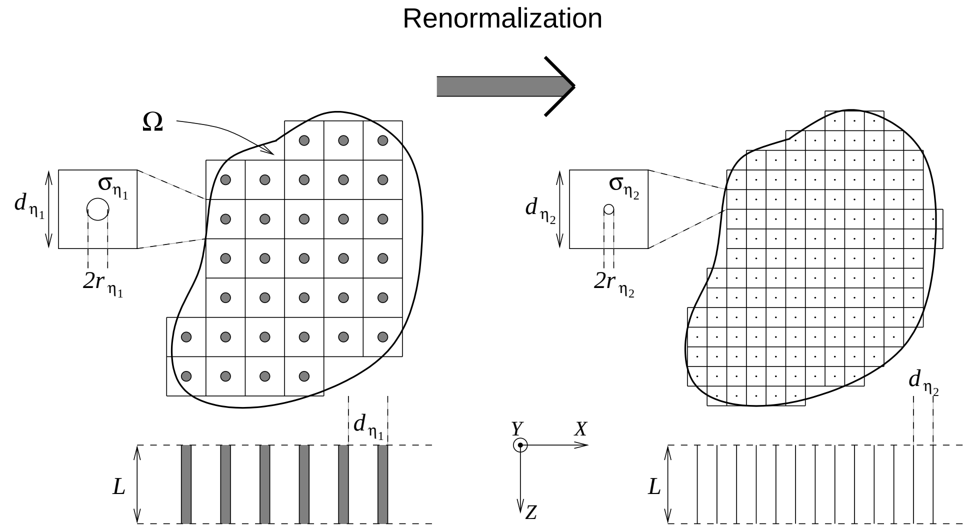

FIGURE 1. The bed-of-nails structure and the renormalization process. The conducting fibers occupy a region $\Omega \subsetneq \mathbb{R}^2$, are oriented in the z direction, and the structure is periodic in the xy plane. Two renormalized structures are shown, corresponding to η_1 and η_2 respectively, with $\eta_1 > \eta_2$, $d_{\eta_1} > d_{\eta_2}$, $\sigma_{\eta_1} < \sigma_{\eta_2}$ and $r_{\eta_1}/d_{\eta_1} > r_{\eta_2}/d_{\eta_2}$ (see Eqs. 1 and (2)). The real physical structure corresponds by definition to $\eta = 1$: $d_{\eta=1} = d$. The length L and the wavelength λ remain fixed, i.e., *we are homogenizing in the xy plane only*.

unique solution of the following system:

$$\begin{cases} \nabla \times E & = i\omega\mu_0 H \\ \nabla \times H & = -i\omega\varepsilon_0 (E + \frac{P}{\varepsilon_0}\hat{z}) \\ \frac{\partial^2 P}{\partial z^2} + \left(k_0^2 + \frac{2i\pi\gamma}{\kappa}\right) P & = -2\pi\gamma\varepsilon_0 E_z,\, z \in [0,L] \\ \frac{\partial P}{\partial z} & = 0,\, z \in \{0,L\} \end{cases} \tag{4}$$

where P corresponds to a polarization density.

We now proceed to solve the homogeneous limit system (4). Since we are dealing with a system with translational invariance, a slab, we can split the problem into two independent polarization cases: TE, where the electric field is in the xy plane, and TM, where the magnetic field is in the xy plane. However, since we are considering thin wires (small volume fraction) the structure will be transparent to TE waves. We therefore only have to consider TM waves. We choose a coordinate system so that the plane of incidence is the xz plane, with angle of incidence θ, in which case our unknowns will be H_y and P_z. The translation invariance allows us to seek solutions of the form:

$$H_y = u(z)e^{i\alpha x} \quad \text{and} \quad P_z = p(z)e^{i\alpha x}$$

with: $\alpha = k_0 \sin\theta$. Inserting these into system (4) we obtain a system of equations for u and p:

$$\begin{cases} u''(z) + \left(k_0^2 - \alpha^2\right) u(z) & = \alpha\omega p(z) \\ p''(z) + \left(k_0^2 + \frac{2i\pi\gamma}{\kappa} - 2\pi\gamma\right) p(z) & = \frac{2\pi\alpha\gamma}{\omega} u(z),\, z \in [0,L] \end{cases} \tag{5}$$

with the important boundary conditions: $p' = 0$ at $z = 0$ and $z = L$, and u and u' continuous everywhere.

To summarize, we are now capable of modeling a structure with a given d, r, σ, L at a given incident field wavelength λ in the following way. We first obtain the two rescaling parameters κ and γ for the given structure using Eqs. (1) and (2) by plugging in $\eta = 1$. Then, we integrate system (5) to obtain the reflection and transmission coefficients in an explicit form.

The 3D full vector simulations of the bed-of-nails structure were done using the Comsol Multiphysics finite element method [6] software package. The periodicity was implemented using Floquet-Bloch conditions [7] in the two periodic directions (x and y), and absorbing Perfectly Matched Layers [8] in the positive and negative z directions. The linearity of the materials in the structure was used to treat the incident field as a localised source within the obstacle, as detailed in Ref. [9].

Figures 2 and 3 show good agreement between the effective medium model and the finite element simulation. Note that the current density behavior near the boundaries differs between the effective medium model and the finite element model. This is due to the fact that in the macroscopic, homogeneous scenario, one speaks of a polarization field obeying Neumann boundary conditions, as discussed above. In the microscopic scenario however, we have a free conductor carrying current induced by an external electric field. Since in our geometry at the given wavelength the capacitance of the wire endpoints is very small, the accumulation of charge will be correspondingly small, leading to an almost continuous

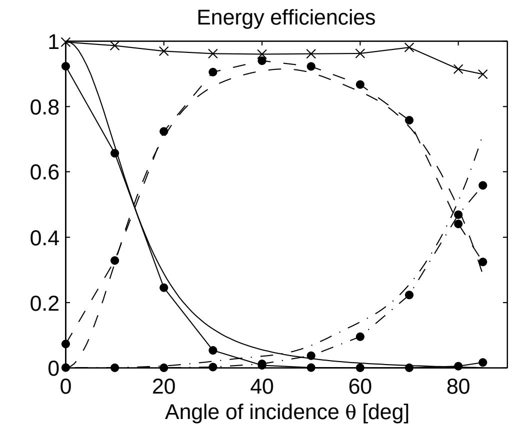

FIGURE 2. Transmission (solid), reflection (dot-dashed) and absorption (dashed) efficiency curves comparing the finite element solution (dot markers) and the and the effective medium solution (no markers) as a function of angle of incidence. The structure has a conductivity $\sigma = 8(\Omega\text{m})^{-1}$, period $d = 0.01$m, and dimensionless parameters $L/d = 120$, $\lambda/d = 20$, $r/d = 0.1$, and $\delta/d = 4.6$. Computational constraints forced us to use a very coarse mesh, which explains the approximate nature of the energy conservation ($\times$ markers) of the finite element model.

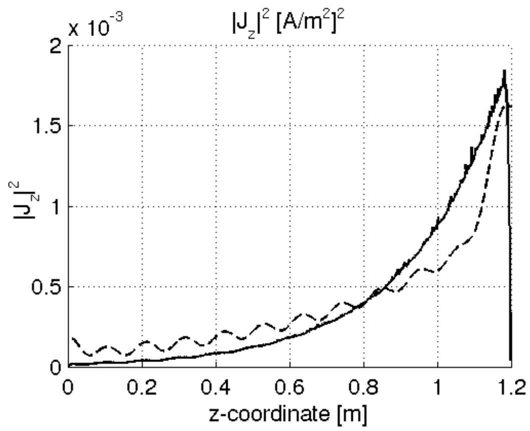

FIGURE 3. Square of the current density for the effective medium solution of Eqs. (5) (dashed) and the finite element solution (solid) as a function of position within the bed-of-nails structure. The structure is the same as in Fig. 2, illuminated at an angle of incidence $\theta = 40°$ from the top.

normal component of the electric field (and therefore also current). Numerically, it seems as if the current goes to zero at the wire endpoints, even though this is not strictly exact. Nevertheless, since in the homogeneous limit the boundary condition of the current is of Neumann type, the convergence of the renormalization process is clearly non-uniform near the boundaries. This provides an explanation for requiring long wires; we want the effect of the boundaries to be small.

It must also be pointed out that the parameters of the particular structure chosen for the illustration in Figs. 2 and 3 were forced upon us by practical constraints: finite element meshing of thin long circular wires requires very large amounts of computer memory and time. Simulation of wires thinner than $r/d = 0.05$ is prohibitive. Consequently, in order to explore a wider domain of the parameter space, our future work will take advantage of the fact that the structures we are interested in have $r \ll d$ and $\delta \gg r$. Such thin conducting structures can be simulated much more efficiently as lines of zero thickness [10] (i.e. *edges*, in the finite element formulation) carrying current and exhibiting an equivalent *linear impedance*. The first results obtained using this approach are excellent using only a fraction of the computing power. It will enable us to model realistic structures that would otherwise be inaccessible.

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